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On diagrams of smooth maps

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In this note we discuss the unfolding theory and the classification of divergent diagrams of smooth map germs $(f,g):\mathbf{R},0 \xleftarrow{f} \mathbf{R}^n,0 \xrightarrow{g} \mathbf{R}^n,0$. The study of general diagrams of smooth mappings from the mapping theoretical view point was proposed by Baas. Dufour and the author studied coposite maps $\to \to \dots \to$ and obtained some satisfactory results. But for diagrams containing cycle (dynamical system) or the divergent part , the basic methods in the singularity theory, i.e. versal unfolding, infinitesimal stability etc fail. For example the C^∞ conjugacy classes of divergent diagrams of smooth map germs are known to have, in general, moduli of infinite dimension called function moduli by Arnold and Dufour. It is seen that multi-germs of divergent diagrams of global maps form very complicated diagrams.

Follwing Izumiya, an ordinary differential equation on $\mathbf{R}^n,0$ is a smooth submanifold S of dim n in the projective cotangent bundle $PT^*\mathbf{R}^n$ and its first integral is a function f on S such that $df \wedge \omega$ is identially zero on S, where ω is the one-form on S induced from that on the cotangent bundle. Izumiya classified some normal forms of the projections $\pi:S \to \mathbf{R}^n$, as map germs, for generic S with the first integral, and also some divergent diagrams $(f,\pi):\mathbf{R} \leftarrow S \to \mathbf{R}^n$ for simple π. Sabbha proved that, after adequate iteration of blow ups φ of the targent space of analytic maps $f:N \to P$ along the locus of those $y \in P$ for which f is not A_f-regular along the fibre $f^{-1}(y)$, the strict transform

$$\begin{array}{cccc} \tilde{f}: & \tilde{N} & \longrightarrow & \tilde{P} \\ & \downarrow & & \downarrow \varphi \\ f: & N & \longrightarrow & P \end{array}$$

is $A_{\tilde{f}}$-reguylar. From the divergent part of this square commutative diagram we can obtain the various topological invariants of infinite dimension for the underlying map f.

In this note we will explane the relation of function moduli and the "contact" class and we show a geometric classification theorem of the contact class of the divergent diagrams. The classification theorem tells that the "contact class" of ordinary differential equations on $R^n,0$ are determined by the diffeo-types of the pairs of the singular point set of the equation and the integral manifold (of codim 1) passing through the origin.

Divergent diagrams $(f_i, g_i): R,0 \leftarrow R^n,0 \to R^n,0$, $i = 1,2$ are L-(left) equivalent if there exist germs of diffeomorphisms φ, ψ of $R^n,0$ such that the diagram commutes

$$\begin{array}{ccccc} R & \xleftarrow{f_1} & R^n,0 & \xrightarrow{g_1} & R^n,0 \\ \| & & \downarrow \varphi & & \downarrow \psi \\ R & \xleftarrow[f_2]{} & R^n,0 & \xrightarrow[g_2]{} & R^n,0 \end{array}$$

An unfolding of (f,g) of dim s is a divergent diagram $(F,G):R \leftarrow R^{n+s},0 \to R^{n+s},0$ such that there exist embeddings φ, ψ of $R^n,0$ into $R^{n+s},0$ such that ψ is transversal to G and the diagram commutes

$$\begin{array}{ccccc} R & \xleftarrow{f} & R^n,0 & \xrightarrow{g} & R^n,0 \\ \| & & \downarrow \varphi & & \downarrow \psi \\ R & \xleftarrow[F]{} & R^{n+s},0 & \xrightarrow[G]{} & R^{n+s},0 \end{array}$$

We denote $(\varphi,\psi):(f,g) \to (F,G)$ and call (φ,ψ) a morphism of an unfolding.

The divergent diagrams (f_i,g_i) are K-(<u>contact-)equivalent</u> if there exists an R-algebra isomorphism φ^*: $\mathcal{E}(n)/g_1 \to \mathcal{E}(n)/g_2$ sending the class of f_1 to that of f_2 , where $\mathcal{E}(n)$ denote the ring of smooth function germs on $R^n,0$. The φ^* is induced from a diffeomorphism φ of $(R^n,0)$ and (f_1,g_1) is L-equivalent to the diagram $(f_1',g_1') = (f_1\varphi , g_1\varphi)$, for which $\langle g_1'\rangle = \langle g_2\rangle$ and $f_1' \equiv f_2$ mod g_2 . In other words (f_i,g_i) are contact equivalent if and only if the restrictions f_i: $g_i^{-1}(0) \to R$ are "right equivalent" as function germs on varieties.

INFINITESIMAL STABILITY

Let $f:R^n,0 \to R,0$, $g:R^n,0 \to R^n,0$ be smooth map germs. Let $\Theta(n)$ be the $\mathcal{E}(n)$-module of germs of smooth vector fields on $R^n,0$, $\Theta_g(n) \subset \Theta(n)$ the $\mathcal{E}(n)$-module via g consisting of lifts of vector fields on $R^n,0$ by g and $tf:\Theta(n) \to \mathcal{E}(n)$ the differential of f. We say the divergent diagram (f,g) is <u>infinitesimally g-stable</u> if tf: $\Theta_g(n) \to \mathcal{E}(n)$ is saurjective.

Assume that g is of finite type: dim $\mathcal{E}(n)/g^*m(n) < \infty$, $m(n) \subset \mathcal{E}(n)$ being the maximal ideal of function germs vanishing at the origin. By Nakayama's lemma and the preparation theorem, we see that (f,g) is infinitesimally g-stable if and only if

$$tf(\theta_g(n)) + g^*m(n)\,\mathcal{E}(n) = \mathcal{E}(n)$$

if and only if tf(non singular lifts) genarates $\mathcal{E}(n)/tf$(singular lifts)$+g^*(n)\mathcal{E}(n)$. Since g is of finite type, lifts of singular vector fields (vanishes at the origin) are singular. origin) are singular. at tor Let $\chi_1,\ldots,\chi_s$ be the generator of the $\mathcal{E}(n)$-module of non singular liftable vector fields. Then g is RL-equivalent to the form $g(x,u) = (g_u(x),u)$, $x \in R^{n-s}$, $u \in R^s$ Let f_0 denote the restriction of f to $R^{n-s} \times 0$. Then (f,g) is infinitesimally g-stable if and only $\partial f/\partial u_j|R^{n-s}\times 0$, $j = 1,\ldots,s$ generate $\mathcal{E}(n-s)/tf_0$(singular lifts)$+g_0^*m(n-s)\mathcal{E}(n-s)$.

THEOREM 1 (Classfication theorem) Let (f_i,g), $i = 1,2$ be infinitesimally g-stable and K-equivalent. Then (f_i,g) are L-equivalent.

Proof. We may assume that g is of the above form and $f_1 \equiv f_2$ mod g. Let f_{i0} denote the restriction of f_i to $R^{n-s} \times 0$. By an easy calculation we obtain

$$tf_{10}(\text{singular lifts}) \equiv tf_{20}(\text{singular lifts}) = K$$

mod $g_0^*m(n-s)\mathcal{E}(n-s)$. Since (f_i,g) are infinitesimally g-stable, $\partial f_i/\partial u_j|R^{n-s}\times 0$, $j = 1,\ldots,s$ generate $\mathcal{E}(n-s)/K+g_0$ for $i = 1$ and 2. It is then easy to join f_1 to f_2 by a family f_t, $1 \le t \le 2$ for which (f_t,g) is infinitesimall g stable. By a standard method of Thom-Mather theory we can prove that the family of diagrams (f_t,g) is C^∞ trivial with respect to the L-equivalence relation and hence (f_1, g) is L-equivalent to (f_2, g). This completes the proof.

FROZEN UNFOLDING (g-UNFOLDING)

A g-unfolding of dim s of a divergent diagram (f,g) is an unfolding of the form

$$(F,G) = (f_u \ , \ (g,u)) \quad , \quad u \in \mathbf{R}^s \ , \ f_0 = f \quad .$$

Let (F',G') be an unfolding of (f,g) and $T_{G'} \subset \mathbf{R}^{n+s}$ be the subspace of non singular liftable vector fields evaluated at the origin. If the embedding ψ for the unfolding is transversal to $T_{G'}$, then (F',G') is L-equivalent to a g-unfolding of the above form.

Conversely let ψ' be an embedding of $\mathbf{R}^n,0$ into $\mathbf{R}^{n+s},0$ transversal to $T_{G'}$ and define the diagram (f'',g'') by the commutative diagram of fibre product

$$\begin{tikzcd}
\mathbf{R} \arrow[d, equal] & \mathbf{R}^{n+s} \arrow[l, "F'"'] \arrow[r, "G'"] & \mathbf{R}^{n+s} \\
\mathbf{R} & \mathbf{R}^n \arrow[l, "f''"] \arrow[r, "g''"'] \arrow[u, "\varphi'"'] & \mathbf{R}^{n+s} \arrow[u, "\varphi'"']
\end{tikzcd}$$

Then (f'',g'') is L-equivalent to an (f',g).

Let $\varphi_1, \ldots, \varphi_s$ be the generators of $\mathcal{E}(n)/tf(\Theta_g(n)) + g^* m(n)\mathcal{E}(n)$ and define g-unfolding by

$$F(x,u) = f(x) + \varphi_1(x)u_1 + \ldots + \varphi_s(x)u_s \quad , \quad x \in \mathbf{R}^n, \ u \in \mathbf{R}^s$$

Then (F,G) is infinitesimally G-stable.

THEOREM 2 (Versality theorem) Let (f',g) be K-equivalent to (f,g). Then there exists a morphism (φ,ψ) of (f',g) into the infinitesimally stable unfolding (F,G) of (f,g) such that ψ is transversal to the subspace $T_G \subset \mathbf{R}^{n+s}$.

Proof. Since (f',g) is K-equivalent to (f,g), (f',g) admits an infinitesimally G-stable g-unfolding (F',G) of the same dimension s. By Theorem 1, (F',G) is L-equivalent to (F,G) by diffeomorphisms φ, ψ respecting T_G (this follows the proof of the theorem). Then the composite $\psi\ i$ in the following diagram of morphisms of unfoldings is transversal to T_G

$$\begin{array}{lcccc}
(F,G): & \mathbf{R} & \longleftarrow & \mathbf{R}^{n+s} & \longrightarrow & \mathbf{R}^{n+s} \\
 & \Vert & & \uparrow \varphi & & \uparrow \psi \\
(F',G): & \mathbf{R} & \longleftarrow & \mathbf{R}^{n+s} & \longrightarrow & \mathbf{R}^{n+s} \\
 & \Vert & & \uparrow & & \uparrow i \\
(f',g): & \mathbf{R} & \longleftarrow & \mathbf{R}^{n} & \longrightarrow & \mathbf{R}^{n}
\end{array}$$

This completes the proof.

Similarly to the above theorem wee obtain

THEOREM 3 Let (f_t,g_t), $t \in \mathbf{R}^a$, $(f_0,g_0) = (f,g)$ be a smooth family of divergent diagram and (F,G) the infinitesimally G stable g-unfolding. Then there exists a family of morphisms of unfolding (φ_t,ψ_t): $(f_t,g_t) \to (F,G)$.

FUNCTION MODULI

THEOREM (Normal form theorem) Let (f',g) be K-equivalent to

(f,g) and $\varphi_1,\dots,\varphi_s$ be the generators of $\mathcal{E}(n)/tf(\Theta_g(n))+g^*m(n)\mathcal{E}(n)$. Assume that $T_g = 0$ i.e. g does not admit non singular lifts. Then (f',g) is L-equivalent to a diagram (f'',g) of the form

$$f''(x) = f(x) + \varphi_1(x)h_1(g(x)) + \dots + \varphi_s(x)h_s(g(x))$$

where $h_1,\dots,h_s$ are smooth functions on the target of g.

Proof. Let (F,G) be the infinitesimally G stable g-unfolding of (f,g) defined in the above. By THEOREM 2 there exists a morphism (φ,ψ): $(f',g) \to (F,G)$, and ψ is transversal to $T_G = 0 \times \mathbf{R}^s$. Represent the image of ψ as a graph of a smooth map h: $\mathbf{R}^n \to \mathbf{R}^s$. Then the image of φ is the graph of the map $h(g)$: $\mathbf{R}^n \to \mathbf{R}^s$ and the composition $f'' = F(h)$ is of the form in the theorem. This completes the proof.

The second term of the above f'' is called <u>function moduli</u> (the parameters are the smooth functions $h_1,\dots,h_s$).

EXAMPLE 1 Let (f,g): $\mathbf{R} \leftarrow \mathbf{R}^2 \to \mathbf{R}^2$, $f(x,y) = y$ and $g(x,y) = (x,\ y^3+xy)$. Then $\mathcal{E}(2)/tf(\text{lifts})+g^*m(2)\mathcal{E}(2)$ is generated by the constant function 1 over $\mathcal{E}(2)$ via g. THerefore diagrams (f',g) K-equivalent to (f,g) is L-equivalent to the normal form

$$f + h(g) = y + h(x,\ y^3+xy)$$

EXAMPLE 2 Let $f(x,y) = x + y^2$ and $g(x,y) = (x,\ y^3+xy^2)$. Then $\mathcal{E}(2)/tf(\text{lifts})+g^*m(2)\mathcal{E}(2)$ is generated by 1 and the normal form for K¥equivalent diagrams (f',g) is

$$x + y^2 + h(x,\ y^3+xy^2)$$

GEOMETRIC CLASSIFICATION

In this section we suppose all map germs are complex analytic. The various notions follows the real smooth case. We assume that the restriction of $g\colon \mathbf{C}^n \to \mathbf{C}^n$ to the critical point set $\Sigma(f)$ is a normalization of the discriminal set: the image, and also $\Sigma(g)$ and $g\Sigma(g)$ are singular at the orogin. Such map germs are classified by the diffeo-types of the discriminant set $g\Sigma(g)$ by Hormander, Wirthmuller etc.

Let (f_i, g_i) be divergent diagrams and assume that the pairs $(g_i(f_i^{-1}(0)),\ g_i\Sigma(g_i))$, $i = 1,2$ are diffeomorphic. Then the restriction of diffeomorphism to the critical point sets extends and lifts to a diffeomorphism φ of the source spaces of g_i sending $f_1^{-1}(0)$ to $f_2^{-1}(0)$ and $g_1^{-1}(0)$ to $g_2^{-1}(0)$ respectively. We say (f_i, g_i) are <u>quasi homogeneous</u> if this comdition implies K-equivalence. For those divergent diagrams, the k-equivalence classes are thus determined by the geometry of the integral manifolds and the singular point sets, and the normal forms are writen in terms of function foduli.